
بسم الله الرحمن الرحيم

# ADVANCED PROBABILITY THEORY

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# REFERENCES


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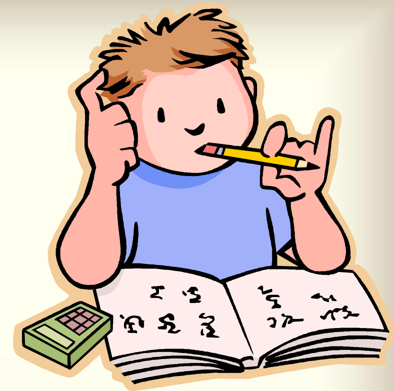
# SILLABUS



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# 1. Markov's Inequality

If $X$ is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

# Exercise 1

Misalkan X berdistribusi seragam diskrit pada interval [1,5]. Tentukan nilai peluang berikut ini secara eksak dan bandingkan dengan nilai batas menggunakan pertidaksamaan Markov:

a. $P(X \geq 3)$

b. $P(X < 3)$

c. $P(X \geq 5)$

d. $P(X < 5)$

# 2. Chebyshev's Inequality

If $X$ is a random variable with finite mean $\mu$ and variance $\sigma^2$, then for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

# Exercise 2

Misalkan X berdistribusi uniform diskrit pada interval [1,5]. Tentukan nilai peluang berikut ini secara eksak dan bandingkan dengan nilai batas menggunakan pertidaksamaan Chebyshev:

a. $P(|X - 3| \geq 1)$

b. $P(|X - 3| < 3)$

# Exercise 3

Misalkan rata-rata hasil produksi dalam sebulan sebesar 50 unit dengan variansi 25 unit, tentukan batas-batas nilai peluang bahwa:

a. hasil produksi dalam suatu bulan akan melebihi 75 unit
b. Hasil produksi dalam suatu bulan akan lebih dari 40 dan kurang dari 60 unit.

# Exercise 4

- Jika variansi X adalah 0, maka tunjukkan bahwa nilai X adalah konstan yaitu rata-ratanya.

# 3. One-Sided Chebyshev Inequality

If $X$ is a random variable with mean 0 and finite variance $\sigma^2$, then, for any $a > 0$,

$$P\{X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

# 4. Chernoff's Inequality

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$$P(X \geq a) \leq e^{-ta} \cdot M(t) \quad , \quad \text{untuk setiap } t > 0$$

$$P(X \leq a) \leq e^{-ta} \cdot M(t) \quad , \quad \text{untuk setiap } t < 0$$

# 5. The Weak Law of Large Number

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, each having finite mean $E[X_i] = \mu$. Then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \quad \text{as} \quad n \to \infty$$

# 6. The Strong Law of Large Number

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with mean $\mu$. Then, the sequence of sample means $M_n = (X_1 + \cdots + X_n)/n$ converges to $\mu$, *with probability 1*, in the sense that

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right) = 1.$$

# 7. The Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables each having mean $\mu$ and variance $\sigma^2$. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx \quad \text{as} \quad n \to \infty$$

**Theorem 9.1 (Central Limit Theorem for Binomial Distributions)** For the binomial distribution $b(n, p, j)$ we have

$$\lim_{n \to \infty} \sqrt{npq}\, b(n, p, \langle np + x\sqrt{npq}\rangle) = \phi(x) ,$$

where $\phi(x)$ is the standard normal density.

The proof of this theorem can be carried out using Stirling's approximation from Section 3.1. We indicate this method of proof by considering the case $x = 0$. In this case, the theorem states that

$$\lim_{n \to \infty} \sqrt{npq}\, b(n, p, \langle np\rangle) = \frac{1}{\sqrt{2\pi}} = .3989\ldots .$$

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In order to simplify the calculation, we assume that $np$ is an integer, so that $\langle np \rangle = np$. Then

$$\sqrt{npq}\, b(n, p, np) = \sqrt{npq}\, p^{np} q^{nq} \frac{n!}{(np)!\,(nq)!} \ .$$

Recall that Stirling's formula (see Theorem 3.3) states that

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \qquad \text{as} \quad n \to \infty \ .$$

Using this, we have

$$\sqrt{npq}\, b(n, p, np) \sim \frac{\sqrt{npq}\, p^{np} q^{nq} \sqrt{2\pi n}\, n^n e^{-n}}{\sqrt{2\pi np}\sqrt{2\pi nq}\, (np)^{np} (nq)^{nq} e^{-np} e^{-nq}} \ ,$$

which simplifies to $1/\sqrt{2\pi}$. $\square$

# Exercise

Jika sebuah koin dilempar sebanyak 100 kali, tentukan secara eksak dan pendekatan normal, bahwa:

1. muncul sisi angka sebanyak 55 kali
2. Muncul sisi gambar antara 40 sampai 60 kali

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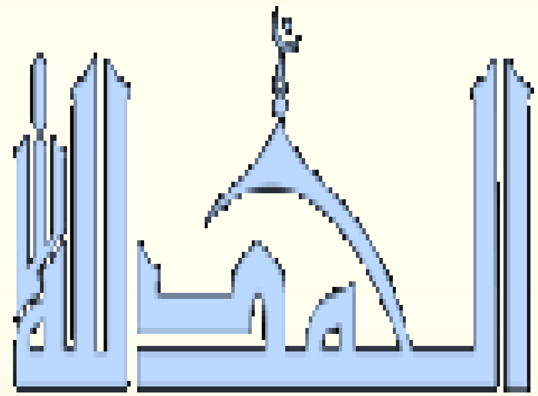


# SELAMAT BELAJAR

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